

OUTER AUTOMORPHISMS OF ALGEBRAIC GROUPS AND DETERMINING GROUPS BY THEIR MAXIMAL TORI

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ABSTRACT. We give a cohomological criterion for existence of outer automorphisms of a semisimple algebraic group over an arbitrary field. This criterion is then applied to the special case of groups of type D_{2n} over a global field, which completes some of the main results from the paper “Weakly commensurable arithmetic groups and isospectral locally symmetric spaces” (Pub. Math. IHES, 2009) by Prasad and Rapinchuk and gives a new proof of a result from another paper by the same authors.

One goal of this paper is the (rather technical) Theorem 17 below, which completes some of the main results in the remarkable paper [PrR09] by Gopal Prasad and Andrei Rapinchuk. For example, combining their Theorem 7.5 with our Theorem 17 gives:

Theorem 1. *Let G_1 and G_2 be connected absolutely simple algebraic groups over a number field K that have the same K -isomorphism classes of maximal K -tori. Then:*

- (1) *G_1 and G_2 have the same Killing-Cartan type (and even the same quasi-split inner form) or one has type B_n and the other has type C_n .*
- (2) *If G_1 and G_2 are isomorphic over an algebraic closure of K and they are not of type A_n for $n \geq 2$, D_{2n+1} , or E_6 , then G_1 and G_2 are K -isomorphic.*

This result is essentially proved by Prasad-Rapinchuk in [PrR09], except that paper omits types D_{2n} for $2n \geq 4$ in (2). Our Theorem 17 gives a new proof of the $2n \geq 6$ case (treated by Prasad-Rapinchuk in a later paper [PrR10, §9]) and settles the last remaining case of groups of type D_4 . Note that in Theorem 1(2), types A_n , D_{2n+1} , and E_6 are genuine exceptions by [PrR09, 7.6].

Similarly, combining our Theorem 17 with the arguments in [PrR09] implies that their Theorems 4, 8.16, and 10.4 remain true if you delete “ D_4 ” from their statements—that is, the conclusions of those theorems regarding weak commensurability, locally symmetric spaces, etc., also hold for groups of type D_4 .

We mention the following specific result as an additional illustration. For a Riemannian manifold M , write $\mathbb{Q}L(M)$ for the set of rational multiples of lengths of closed geodesics of M .

Theorem 2. *Let M_1 and M_2 be arithmetic quotients of real hyperbolic space \mathbf{H}^n for some $n \not\equiv 1 \pmod{4}$. If $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$, then M_1 and M_2 are commensurable (i.e., M_1 and M_2 have a common finite-sheeted cover).*

The converse holds with no restriction on n , see [PrR09, Cor. 8.7]. The theorem itself holds for $n = 2$ by [R]; for $n = 3$ by [CHLR]; and for $n = 4, 6, 8, \dots$ and $n =$

11, 15, 19, ... by [PrR09, Cor. 8.17] (which relies on [PrR10]). The last remaining case, $n = 7$, follows from Theorem 17 below and arguments as in [PrR09]. The conclusion of Theorem 2 is false for $n = 5, 9, 13, \dots$ by Construction 9.15 in [PrR09].

The other goal of this paper is Theorem 9, which addresses the more general setting of a semisimple algebraic group G over an arbitrary field k . That theorem gives a cohomological criterion for the existence of outer automorphisms of G , i.e., for the existence of k -points on non-identity components of $\text{Aut}(G)$. This criterion and the examples we give of when it holds make up the bulk of the proof of Theorem 17, which concerns groups over global fields.

Notation. A *global field* is a finite extension of \mathbb{Q} or $\mathbb{F}_p(t)$ for some prime p . A (non-archimedean) *local field* is the completion of a global field with respect to a discrete valuation, i.e., a finite extension of \mathbb{Q}_p or $\mathbb{F}_p((t))$ for some prime p .

We write $H^d(k, G)$ for the d -th flat (fppf) cohomology set $H^d(\text{Spec } k, G)$ when G is an algebraic affine group scheme over a field k . In case G is smooth, it is the same as the Galois cohomology set $H^d(\text{Gal}(k), G(k_{\text{sep}}))$ where k_{sep} denotes a separable closure of k and $\text{Gal}(k)$ denotes the group of k -automorphisms of k_{sep} .

We refer to [PIR], [Sp98], and [KMRT] for general background on semisimple algebraic groups. Such a group G is an *inner form* of G' if there is a class $\gamma \in H^1(k, \overline{G})$, for \overline{G} the adjoint group of G , such that G' is isomorphic to G twisted by γ . We write G_γ for the group G twisted by the cocycle γ , following the \TeX -friendly notation of [KMRT, p. 387] instead of Serre's more logical ${}_\gamma G$. We say simply that G is *inner* or *of inner type* if it is an inner form of a split group; if G is not inner then it is *outer*.

For a group scheme D of multiplicative type, we put D^* for its dual $\text{Hom}(D, \mathbb{G}_m)$.

1. BACKGROUND: THE TITS ALGEBRAS DETERMINE THE TITS CLASS

Fix a semisimple algebraic group G over a field k . Its simply connected cover \tilde{G} and adjoint group \overline{G} fit into an exact sequence

$$(3) \quad 1 \longrightarrow Z \longrightarrow \tilde{G} \longrightarrow \overline{G} \longrightarrow 1$$

where Z denotes the (scheme-theoretic) center of \tilde{G} . Write $\delta: H^1(k, \overline{G}) \rightarrow H^2(k, Z)$ for the corresponding coboundary map.

There is a unique element $\nu_G \in H^1(k, \overline{G})$ such that the twisted group \overline{G}_{ν_G} is quasi-split [KMRT, 31.6], and the *Tits class* t_G of G is defined to be $t_G := -\delta(\nu_G) \in H^2(k, Z)$. The element t_G depends only on the isogeny class of G .

For $\gamma \in H^1(k, \overline{G})$, the center of the twisted group \tilde{G}_γ is naturally identified with (and not merely isomorphic to) Z , and a standard twisting argument shows that

$$(4) \quad t_{G_\gamma} = t_G + \delta(\gamma).$$

Example 5. If G itself is quasi-split, then $t_G = 0$ and for every $\gamma \in H^1(k, \overline{G})$ we have $t_{G_\gamma} = \delta(\gamma)$.

Definition 6 (Tits algebras). A *Tits algebra* of G is an element

$$\chi(t_G) \in H^2(k(\chi), \mathbb{G}_m) \quad \text{for } \chi \in Z^*,$$

where $k(\chi)$ denotes the subfield of k_{sep} of elements fixed by the stabilizer of χ in $\text{Gal}(k)$, i.e., $k(\chi)$ is the smallest separable extension of k so that χ is fixed by $\text{Gal}(k(\chi))$.

We can modify this definition to replace elements of Z^* with weights. Fix a pinning for \tilde{G} over k_{sep} involving a maximal k -torus \tilde{T} . As Z is contained in \tilde{T} , every weight λ —i.e., every $\lambda \in \tilde{T}^*$ —induces by restriction an element of Z^* and we define $\lambda(t_G)$ to be $\lambda|_Z(t_G)$.

Regarding history, the Tits algebras of G were defined in [T]. The class $\lambda(t_G)$ measures the failure of the irreducible representation of \tilde{G} with highest weight λ —which is defined over k_{sep} —to be defined over k . Roughly speaking, a typical example of a Tits algebra is provided by the even Clifford algebra of the special orthogonal group of a quadratic form, see for example [KMRT, §27].

Obviously, the Tits class t_G determines the Tits algebras $\chi(t_G)$. The converse also holds:

Proposition 7. *If G is absolutely almost simple, then the natural map*

$$\prod \lambda: H^2(k, Z) \rightarrow \prod H^2(k(\lambda|_Z), \mathbb{G}_m)$$

is injective, where the products range over minuscule weights λ .

This proposition can probably be viewed as folklore. I learned it from Alexander Merkurjev and Anne Quéguiner. Below, we will use the following restatement: Given classes $\gamma_1, \gamma_2 \in H^1(k, \tilde{G})$, if $\lambda(\delta(\gamma_1)) = \lambda(\delta(\gamma_2))$ for every minuscule weight λ of G , then $\delta(\gamma_1) = \delta(\gamma_2)$.

Proof of Proposition 7. As \tilde{G} is simply connected, the weight lattice P in the pinning is identified with \tilde{T} and the root lattice Q is the kernel of the restriction $\tilde{T}^* \rightarrow Z^*$. Since G is assumed absolutely almost simple, its root system is irreducible, and this surjection identifies the minuscule (dominant) weights with the nonzero elements of Z^* [B, §VI.2, Exercise 5a]. Therefore, the claim is equivalent to showing that the map

$$(8) \quad \prod_{\chi \in Z^*} \chi: H^2(k, Z) \rightarrow \prod_{\chi \in Z^*} H^2(k(\chi), Z)$$

is injective. This claim depends only on Z , so we may replace G with G_{ν_G} and so assume that G is quasi-split.

We choose the maximal k -torus \tilde{T} and pinning in \tilde{G} so that the usual Galois action preserves the fundamental chamber, i.e., permutes the set of simple roots Δ and the fundamental dominant weights. In the short exact sequence $1 \rightarrow Z \rightarrow \tilde{T} \rightarrow \tilde{T}/Z \rightarrow 1$, the set Δ is a basis for the lattice $(\tilde{T}/Z)^*$, so $H^1(k, \tilde{T}/Z)$ is zero and the map $H^2(k, Z) \rightarrow H^2(k, \tilde{T})$ is injective.

We fix a set S of representatives of the $\text{Gal}(k)$ -orbits in Δ and write α_s (resp., λ_s) for the simple root (resp., fundamental dominant weight) corresponding to $s \in S$. Because \tilde{G} is simply connected,

$$\tilde{T} \cong \prod_{s \in S} R_{k(\lambda_s|_Z)/k}(\text{im } h_{\alpha_s})$$

where h_{α_s} denotes the homomorphism $\mathbb{G}_m \rightarrow \tilde{T}$ corresponding to the coroot α_s^\vee [St, p. 44, Cor.]—note that the Weil restriction term makes sense because the stabilizers of α_s , λ_s , and $\lambda_s|_Z$ all agree because of our particular choice of pinning. The inclusion of Z in \tilde{T} amounts to the product $\prod_{s \in S} \lambda_s$, hence (8) is injective and the claim is proved. \square

2. OUTER AUTOMORPHISMS OF SEMISIMPLE GROUPS

We maintain the notation of the previous section, so that G is semisimple over a field k and Δ is a set of simple roots, equivalently, the Dynkin diagram of G . The Galois action on Δ induces an action on $\text{Aut}(\Delta)$, and in this way we view $\text{Aut}(\Delta)$ as a finite étale (but not necessarily constant) group scheme. There is a map

$$\alpha: \text{Aut}(G)(k) \rightarrow \text{Aut}(\Delta)(k)$$

described for example in [Sp 98, §16.3], and one can ask if this map is surjective. That is, does every connected component of $\text{Aut}(G) \times k_{\text{sep}}$ that is defined over k necessarily have a k -point?

One obstruction to α being surjective can come from the fundamental group, so we assume that G is simply connected. (One could equivalently assume that G is adjoint.) Another obstruction comes from the Tits class, as we now explain. There is a commutative diagram

$$\begin{array}{ccc} \text{Aut}(G) & \xrightarrow{\alpha} & \text{Aut}(\Delta) \\ \downarrow & \swarrow & \\ \text{Aut}(Z) & & \end{array}$$

where the diagonal arrow comes from the natural action of $\text{Aut}(\Delta)$ on the coroot lattice. Hence $\text{Aut}(\Delta)(k)$ acts on $H^2(k, Z)$ and we have:

Theorem 9. *Recall that G is assumed semisimple and simply connected. Then there is an inclusion*

$$(10) \quad \text{im}[\alpha: \text{Aut}(G)(k) \rightarrow \text{Aut}(\Delta)(k)] \subseteq \{\pi \in \text{Aut}(\Delta)(k) \mid \pi(t_G) = t_G\}.$$

Furthermore, the following are equivalent:

- (a) Equality holds in (10).
- (b) The sequence $H^1(k, Z) \rightarrow H^1(k, G) \rightarrow H^1(k, \text{Aut}(G))$ is exact.
- (c) $\ker \delta \cap \ker [H^1(k, \overline{G}) \rightarrow H^1(k, \text{Aut}(\overline{G}))] = 0$.

Proof. We consider the interlocking exact sequences

$$\begin{array}{ccccccc} & & & & H^1(k, Z) & & \\ & & & & \downarrow & & \\ & & & & H^1(k, G) & & \\ & & & & \downarrow q & & \\ \text{Aut}(G)(k) & \xrightarrow{\alpha} & \text{Aut}(\Delta)(k) & \xrightarrow{\beta} & H^1(k, \overline{G}) & \xrightarrow{\varepsilon} & H^1(k, \text{Aut}(G)) \\ & & & & \downarrow \delta & & \\ & & & & H^2(k, Z) & & \end{array}$$

The crux is to prove that

$$(11) \quad \pi(t_{G_{\beta(\pi)}}) = t_G \quad \text{for } \pi \in \text{Aut}(\Delta)(k).$$

Since \overline{G} and $\text{Aut}(G)$ are smooth, we may view their corresponding H^1 's as Galois cohomology. Put $\gamma := \beta(\pi)$, so $\gamma_\sigma = f^{-1} \sigma f$ for some $f \in \text{Aut}(G)(k_{\text{sep}})$ and every

The map f gives a k -isomorphism $G_\gamma \xrightarrow{\sim} G$. Sequence (3) gives a commutative diagram

$$\begin{array}{ccc} H^1(k, \overline{G}_\gamma) & \xrightarrow{\delta_\gamma} & H^2(k, Z) \\ f \downarrow & & f \downarrow \\ H^1(k, \overline{G}) & \xrightarrow{\delta} & H^2(k, Z). \end{array}$$

It follows that $\pi \in \text{Aut}(\Delta)(k)$ satisfies $\pi(t_G) = t_G$ if and only if $t_{G_{\beta(\pi)}} = t_G$, if and only if $\delta(\beta(\pi)) = 0$. That is, in (10), the left side is $\ker \beta$ and the right side is $\ker \delta\beta$, which makes the inclusion in (10) and the equivalence of (a) and (c) obvious. Statement (b) says that $\ker \varepsilon q = \ker q$, i.e., $\ker \varepsilon \cap \text{im } q = 0$, which is (c). \square

Example 12. Fix a prime p and write x_1, x_2 for the two square roots of p in $k := \mathbb{Q}(\sqrt{p})$. For $i = 1, 2$, let H_i be the group of type G_2 associated with the 3-Pfister quadratic form $\phi_i := \langle\langle -1, -1, x_i \rangle\rangle$. For $G = H_1 \times H_2$, the Tits index is



Nonetheless, we now list “many” cases in which equality holds in (10).

Example 14. If $H^1(k, G) = 0$, then trivially Th. 9(b) holds. That is, (a)–(c) hold for every semisimple simply connected G if k is local (by Kneser-Bruhat-Tits), global with no real embeddings (Kneser-Harder-Chernousov), or the function field of a complex surface (de Jong-He-Starr-Gille), and conjecturally if the cohomological dimension of k is at most 2 (Serre).

(i) *if G is of inner type.* If G is of inner type A ($n \geq 2$), then $\text{Aut}(\Delta) = \mathbb{Z}/2\mathbb{Z}$ and the nontrivial element π acts via $z \mapsto z^{-1}$ on Z , hence $\pi(t_G) = -t_G$. If $2t_G = 0$, then G is $\text{SL}_1(D)$ for D a central simple algebra of degree $n+1$ such that there is an anti-automorphism σ of D , hence $g \mapsto \sigma(g)^{-1}$ is a k -automorphism of G mapping to π . (By a theorem of Albert [Sch, Th. 8.8.4]

one can even arrange for σ to have order 2, hence for this automorphism of G to have order 2.)

Next let G be of type 1D_n for $n \geq 5$ and suppose that the nonidentity element $\pi \in \text{Aut}(\Delta)(k)$ fixes the Tits class t_G . The group G is isomorphic to $\text{Spin}(A, \sigma, f)$ for some central simple k -algebra A of degree $2n$ and quadratic pair (σ, f) on A such that the even Clifford algebra $C_0(A, \sigma, f)$ is isomorphic to a direct product $C_+ \times C_-$ of central simple algebras. Since π fixes the Tits class, the algebras C_+ and C_- are isomorphic. The equation $[A] + [C_+] - [C_-] = 0$ holds in the Brauer group of k by [KMRT, 9.12] (alternatively, as a consequence of the fact that the cocenter is an abelian group of order 4). Therefore, A is split. Let $\phi \in \mathcal{O}(A, \sigma, f)(k)$ be a hyperplane reflection as in [KMRT, 12.13]; it does not lie in the identity component of $\mathcal{O}(A, \sigma, f)$. The automorphism of $\text{SO}(A, \sigma, f)$ given by $g \mapsto \phi g \phi^{-1}$ lifts to an automorphism of $\text{Spin}(A, \sigma, f)$ that is outer, i.e., that induces the automorphism π on Δ . (To recap: given a nonzero $\pi \in \text{Aut}(\Delta)(k)$ that preserves the Tits class, we deduced that A is split and therefore (A, σ, f) has an improper isometry. Conversely, Lemma 1b from [Kn, p. 42] shows: if $\text{char } k \neq 2$ and (A, σ) has an improper isometry, then A is split and obviously such a π exists.)

For the remaining cases, we point out merely that an outer automorphism of order 3 in the D_4 case exists when $t_G = 0$ by triality [SpV, 3.6.3, 3.6.4] and an outer automorphism of order 2 in the E_6 case when $t_G = 0$ is provided by the “standard automorphism” of a J -structure [Sp73, p. 150].

- (ii) if G is the special unitary group of a hermitian form relative to a separable quadratic extension K/k , i.e., G is of type 2A_n and $\text{res}_{K/k}(t_G)$ is zero in $H^2(K, Z)$. We leave the details in this case as an exercise.
- (iii) if k is real closed. By the above cases, we may assume that G has type 2D_n (for $n \geq 4$) or 2E_6 . In the first case, $\text{Aut}(\Delta)(k) = \mathbb{Z}/2\mathbb{Z}$ (also for $n = 4$) and G is the spin group of a quadratic form by [KMRT, 9.14], so a hyperplane reflection gives the desired k -automorphism.

In case G has type 2E_6 , combining pages 37, 38, 119, and 120 in [J] shows that the (outer) automorphism of the Lie algebra Jacobson denotes by t is defined over k .

I don’t know any examples of absolutely almost simple G where conditions (a)–(c) fail. Furthermore, in all of the examples above, every π from the right side of (10) is not only of the form $\alpha(f)$ for some $f \in \text{Aut}(G)(k)$, but one can even pick f to have the same order as π .

3. GROUPS OF TYPE D_{even} OVER LOCAL FIELDS

The main point of this section is to prove the following lemma.

Lemma 16. *Let G be an adjoint semisimple group over a field k , and fix a maximal k -torus T in G . If z_1, z_2 are in the image of the map $H^1(k, T) \rightarrow H^1(k, G)$ such that*

- (1) G_{z_1} and G_{z_2} are both quasi-split; or
- (2) T contains a maximal k -split torus in both G_{z_1} and G_{z_2} and
 - (a) k is real closed, or
 - (b) k is a (non-archimedean) local field and G has type D_{2n} for some $n \geq 2$,

then $z_1 = z_2$.

Proof. For short, we write G_i for G_{z_i} . In case (1), the uniqueness of the class $\nu_G \in H^1(k, G)$ such that G_{ν_G} is quasi-split (already used in §1) gives that $z_1 = \nu_G = z_2$. So suppose (2) holds. As T is contained in both these groups, their Tits indexes are naturally identified over k . In particular, if one is quasi-split then so is the other, and we are done as in (1). So we assume that neither group is quasi-split.

In case (2a), where k is real closed, one immediately reduces to the case where G is absolutely simple. That case is trivial because the isomorphism class of an adjoint simple group is determined by its Tits index, so G_1 is isomorphic to G_2 . The Tits index also determines the Tits algebras—see pages 211 and 212 of [T] for a recipe—so by Prop. 7, $\delta(z_1) = \delta(z_2)$. The claim now follows from Example 15(iii) and Theorem 9(c).

So assume for the remainder of the proof that (2b) holds. In particular, δ is injective. Number the simple roots of G_1 with respect to T as in [B]. If G_1 has type 2D_4 , we take α_1 to be the root at the end of the Galois-fixed arm of the Tits index. Otherwise, we assign the numbering arbitrarily in case there is ambiguity (e.g., α_{2n-1} and α_{2n}). Note that G_1 cannot have type 3D_4 or 6D_4 , because it is not quasi-split.

As $2\omega_i$ is in the root lattice for every i , the Tits algebras $\omega_i(t_{G_1})$ for $i = 2n-1, 2n$ define up to k -isomorphism a quaternion (Azumaya) algebra D over a quadratic étale k -algebra ℓ . By the exceptional isomorphism $D_2 = A_1 \times A_1$ and a Tits algebra computation, $\mathrm{PGL}_1(D)$ is isomorphic to $\mathrm{PSO}(M_2(H), \sigma, f)$ for H the quaternion algebra underlying $\omega_1(t_{G_1})$ and some quadratic pair (σ, f) such that the even Clifford algebra $C_0(\sigma, f)$ is isomorphic to D , cf. [KMRT, 15.9]. Appending $2n-2$ hyperbolic planes to (σ, f) , we obtain a quadratic pair (σ_0, f_0) such that $C_0(\sigma_0, f_0)$ is Brauer-equivalent to D . As $\mathrm{PSO}(M_{2n}(H), \sigma_0, f_0)$ has the same Tits algebras as G_1 (up to renumbering the simple roots of G_1), Prop. 7 and injectivity of δ implies that the two groups are isomorphic. (We have just given a characteristic-free proof of Tsukamoto's theorem [Sch, 10.3.6], relying on the Bruhat-Tits result that δ is injective.)

Now both G_1 and G_2 have the same Tits index and semisimple anisotropic kernels of Killing-Cartan type a product of A_1 's. As there is a unique quaternion division algebra over each finite extension of k , it follows that G_1 and G_2 have the same Tits class, i.e., $\delta(z_1) = \delta(z_2)$. \square

In the statement of (2b), we cannot replace “ D_{2n} for some $n \geq 2$ ” with “ D_ℓ for some ℓ ” because the claim fails for groups of type D_{odd} . This can be seen already for type $D_3 = A_3$: one can find $z_1, z_2 \in H^1(k, \mathrm{PGL}_4)$ so that G_1 and G_2 are both isomorphic to $\mathrm{Aut}(B)^\circ$ for a division algebra B of degree 4, but $\delta(z_1) = -\delta(z_2)$ in $H^2(k, \mu_4) = \mathbb{Z}/4\mathbb{Z}$. Adding hyperbolic planes as in the proof of the lemma gives a counterexample for all odd ℓ . This counterexample is visible in the proof: for groups G_1, G_2 of type D_ℓ with ℓ odd and ≥ 3 , the semisimple anisotropic kernels have Killing-Cartan type a product of A_1 's and an A_3 and the very last sentence of the proof fails.

4. GROUPS OF TYPE D_{even} OVER GLOBAL FIELDS

The following technical theorem concerning groups over a global field connects our Theorem 9 (about groups over an arbitrary field) with the results in [PrR09]. It implies Theorem 9.1 of [PrR10].

Theorem 17. *Let G_1 and G_2 be adjoint groups of type D_{2n} for some $n \geq 2$ over a global field K , such that G_1 and G_2 have the same quasi-split inner form—i.e., the smallest Galois extension of K over which G_1 is of inner type is the same as for G_2 . If there exists a maximal torus T_i in G_i for $i = 1$ and 2 such that*

- (1) *there is a K_{sep} -isomorphism $\phi : G_1 \rightarrow G_2$ whose restriction to T_1 is a K -isomorphism $T_1 \rightarrow T_2$; and*
- (2) *there is a finite set V of places of K such that:*
 - (a) *For all $v \notin V$, G_1 and G_2 are quasi-split over K_v .*
 - (b) *For all $v \in V$, $(T_i)_{K_v}$ contains a maximal K_v -split torus of $(G_i)_{K_v}$;*

then G_1 and G_2 are isomorphic over K .

The hypotheses are what one obtains by assuming the existence of weakly commensurable arithmetic subgroups, see for example Theorems 1 and 6 and Remark 4.4—and especially p.156—in [PrR09]. Note that the groups appearing in the theorem can be triality, i.e., of type 3D_4 or 6D_4 . We remark that Bruce Allison gave an isomorphism criterion with very different hypotheses in [A, Th. 7.7].

Proof. Write G for the unique adjoint quasi-split group that is an inner form of G_1 and G_2 . By Steinberg [PIR, pp. 338, 339], there is a K_{sep} -isomorphism $\psi_2 : G_2 \rightarrow G$ whose restriction to T_2 is defined over K . We put $\psi_1 := \psi_2 \phi$ and $T := \psi_2(T_2) = \psi_1(T_1)$. Then G_i is isomorphic to G twisted by the 1-cocycle $\sigma \mapsto \psi_i({}^\sigma \psi_i)^{-1}$. But this 1-cocycle consists of elements of $\text{Aut}(G)$ that fix T elementwise, hence belong to T itself. That is, for $i = 1, 2$, there is a cocycle z_i in the image of $H^1(K, T) \rightarrow H^1(K, G)$ such that G_i is isomorphic to G twisted by z_i .¹

Now Lemma 16 gives that $\text{res}_{K_v/K}(z_1) = \text{res}_{K_v/K}(z_2)$ for every v , hence $z_1 = z_2$ by the Kneser-Harder Hasse Principle [PIR, p. 336, Th. 6.22] and G_1 is isomorphic to G_2 over K . \square

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¹This argument does not use the fact that K is a number field nor that G_1 and G_2 have type D_{2n} , so roughly speaking it applies generally to the situation where G_1 and G_2 share a maximal torus over the base field—more precisely, to the situation arising in Remark 4.4 of [PrR09].

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